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Some Geometric Consequences of a Game Theoretic Result

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Players I and II play the following game. From a compact Hausdorff space S, player II secretly selects a point x. A number $i \in \{1, 2, 3,..., p\}$ is secretly chosen by player I. After their selections I receives from II an amount $f_i(x)$ where $f_1, f_2,..., f_p$ are real continuous functions on S. For such a game the following theorem is classical in game theory [2] (Theorem 2.4.2, page 49)

THEOREM. *There exist probability vectors* $\lambda = (\lambda_1, \lambda_2,..., \lambda_p)$ *and* $\mu = (\mu_1, \mu_2,..., \mu_r)$, $r \leqslant p$ *and a constant* v, *such that, by selecting* i *with probability* λ_i *player* I *gets on the average*

$$\sum_i \lambda_i f_i(x) \geqslant v \qquad \text{for any choice } x \in S \text{ of player II.} \tag{1}$$

Also it is possible for player II *to select* r *points* $x_1, x_2,..., x_r$ *with probability* $\mu_1, \mu_2,..., \mu_r$ *so that he loses on the average at most*

$$\sum_j \mu_j f_i(x_j) \leqslant v \qquad \text{for any choice } i \text{ of player I.} \tag{2}$$

Further v *is unique.* v *is called the value of the game. In case* $r = 1$, *namely if* $f_i(x_0) \leqslant v$ *for all* $i = 1, 2,..., p$, *we say that* x_0 *is a good point for player* II.

Using this theorem we will prove the following theorems. The idea is that one can hope to prove similar classes of theorems by viewing them as two-person games. For the sake of completeness we sketch the proof of this game theoretic result.

Proof. Let $\tau: x \to (f_1(x), f_2(x),..., f_p(x))$ be the continuous map from S to R^p. $\tau(S)$ is compact and hence the convex hull K of $\tau(S)$ is compact. For each $y \in K$, let $\Psi(y)$ be the maximum coordinate of y and $v = \min_{y \in K} \Psi(y)$. Let G be the open convex set with all coordinates $< v$ in R^p. $G \cap K = \phi$ and by separation theorem there exists a nontrivial vector λ in R^p and a scalar α with $\langle \lambda, y \rangle \geqslant \alpha$ for $y \in K$ and $\langle \lambda, y \rangle \leqslant \alpha$ for $y \in \bar{G}$. One can show that

λ can be chosen as a probability vector and that $\alpha = v$. Also if $\Psi(y_0) = v$, y_0 is a boundary point of K and also of G and that $y_0 = \sum_{j=1}^{p} \mu_j y^{(j)}$ by Caratheodory's theorem [5] where $y^{(j)} \in \tau(S)$ for all j. Here $\mu = (\mu_1, \mu_2, ..., \mu_p)$ is a probability vector. Since each coordinate of y_0 is at most v, the theorem follows.

Now we prove the following theorems.

THEOREM 1. *Let E be a metric space and $C_1, C_2, ..., C_p$ be closed sets in E whose union is a compact set S. Let every $(p - 1)$ of them have nonnull intersection. Then the following are equivalent.*

(i) $\bigcap_{i=1}^{p} C_i \neq \phi$;

(ii) *the value v of the above game with $f_i(x) = d(x, C_i)$, $x \in S$ is zero (here d denotes the distance)*;

(iii) *there exists a good point for player* II *for the game $f_i(x) = d(x, C_i)$, $x \in S$.*

Proof. (i) $\Rightarrow$ (ii). Let $y \in \bigcap_1^p C_i$. Then from inequalities (1) and (2) we get $0 = \sum \lambda_i d(y, C_i) \geqslant v \geqslant 0$. Thus $v = 0$ and y is a good point for player II.

(ii) $\Rightarrow$ (iii) let $v = 0$. From inequalities (2) we have

$$0 \leqslant \sum_{j=1}^{n} \mu_j \, d(x_j, C_i) \leqslant 0, \qquad i = 1, 2, ..., p.$$

Thus $\mu_j d(x_j, C_i) = 0$ for all i, j. Since $\sum \mu_j = 1$ we have for some j_0, $d(x_{j_0}, C_i) = 0$ $i = 1, 2, ..., p$. The point x_{j_0} is a good point for player II.

(iii) $\Rightarrow$ (i). Let player II have a good point x_0. Then we have two cases. In case some $\lambda_i = 0$, say $\lambda_1 = 0$, then by choosing $y_1 \in \bigcap_{i \neq 1} C_i$ (which exists by assumption) we have $0 = \sum \lambda_i d(y_1, C_i) \geqslant v$. Thus $v \leqslant 0$. This shows $0 \leqslant d(x_0, C_i) \leqslant 0$ for all i and that $x_0 \in \bigcap_1^p C_i$. In case all the λ_i's are positive, at the good point x_0 we have $d(x_0, C_i) = v$ for all i; for otherwise if $d(x_0, C_i) < v$ for some i, then $\sum \lambda_i d(x_0, C_i) < v$ which contradicts (1). Further since $\bigcup_1^p C_i = S$, $x_0 \in C_i$ for some i and that $d(x_0, C_i) = v = 0$ for all i. Thus $x_0 \in \bigcap_1^p C_i$. This completes the proof of the theorem.

THEOREM 2 (Berge). *Let E be a topological vector space and let $C_1, C_2, ..., C_p$ be closed convex sets in E whose union S is compact and convex. If every $(p - 1)$ of them have nonnull intersection, then they all have a point in common.*

Proof. Let $y_j \in \bigcap_{i \neq j} C_i$ $j = 1, 2, ..., p$. In the finite dimensional subspace F containing $y_1, y_2, ..., y_p$ the convex hull T of $\{y_1, y_2, ..., y_p\}$ is the continuous image of probability vectors in R^p and hence compact. Since $C_i \cap F$ are relatively closed, $K_i = C_i \cap T$ are compact in F and further $T = \cup K_i$. Also we have $\bigcap_{i \neq j} K_i \neq \phi, j = 1, 2,..., p$. We can give a norm topology in F which incidentally is the strongest topological vector space topology in F [6]. Further T is compact and since $C_i \cap F$ are also closed in the new norm topology, K_i are compact in F in the norm topology of F. Let $\| \cdot \|$ denote the norm. For the game with payoff $d(x, K_i)$, $x \in T$ we have

$$\sum \mu_j \, d(x_j, K_i) = \sum \mu_j \| x_j - y_{ij} \| \leqslant v \qquad \text{for all } i.$$

Here $x_j \in T$, $y_{ij} \in K_i$ for all i, j. Thus $\| \sum \mu_j x_j - \sum \mu_j y_{ij} \| \leqslant v$ for all i and that for $x_0 = \sum \mu_j x_j \in T$ we have $d(x_0, K_i) \leqslant v$ for all i. Thus x_0 is a good point for player II and from Theorem 1 $\bigcap_1^p K_i \neq \phi$. Thus $\bigcap_1^p C_i \neq \phi$. This completes the proof of the theorem.

Remark. It is well-known that Helly's theorem [5] follows from the above theorem of Berge.

THEOREM 3. *Let $C_1, C_2,..., C_p$ be any p compact convex sets in a normed linear space E. Let every $(p - 1)$ of them have nonnull intersection, but all of them have empty intersection. Then there exists a sphere of radius v that touches all the C_i's. Further no sphere of radius $< v$ touches all of them with a center in the convex hull of $\bigcup_1^p C_i$. In case E is a Hilbert space, no sphere with radius $< v$ and with a center anywhere in E could intersect all the C_i's.*

Proof. Consider the usual game with payoff $f_i(x) = d(x, C_i)$ where $x \in$ convex hull of $\cup C_i = S$. In case some $\lambda_i = 0$ in inequality (1), we know that $\bigcap_1^p C_i \neq \phi$ by Theorem 1. Since by assumption $\bigcap_1^p C_i = \phi$, we have all the λ_i's positive for player I. Thus we have

$$\sum_j \mu_j \, d(x_j, C_i) = \sum \mu_j \| x_j - y_{ij} \| = v \qquad \text{for all } i,$$

where $x_j \in T$ and $y_{ij} \in C_i$ for all i, j.

Again as in Theorem 2

$$\left\| \sum \mu_j x_j - \sum \mu_j y_{ij} \right\| \leqslant v$$

and that $x_0 = \sum \mu_j x_j \in S$ is a good point for player II. As before (see the proof of (iii) $\Rightarrow$ (i) in Theorem 1) since all λ_i's are positive $d(x_0, C_i) = v$ for all i. Namely a sphere with center x_0 and radius v intersects all the sets $C_1, C_2,..., C_p$. Further $v \neq 0$, for otherwise by (ii) $\Rightarrow$ (i) in Theorem 1 we will contradict $\bigcap_1^p C_i = \phi$. Also no sphere of radius smaller then v and with a center in S could touch all the C_i's for in such a case we will have $d(y, C_i) < v$

for some y in S which contradicts the fact that v is the value of the game. Finally to check the last assertion in the theorem let us assume without loss of generality that $0 \notin S$ and a sphere of radius $r < v$ with center 0, intersect all C_i's. Since we assume E to be a Hilbert space, there exists a unique point x_0 which has the minimum norm among all points of S. Let y be any point of S. Consider $\lambda y + (1 - \lambda) x_0 \in S$ for any $0 \leqslant \lambda \leqslant 1$. We have

$$\begin{aligned} \| \lambda y + (1 - \lambda) x_0 \|^2 &= \| x_0 + \lambda(y - x_0)\|^2 \\ &= \| x_0 \|^2 + 2\lambda\langle y - x_0 , x_0\rangle + \lambda^2 \| y - x_0 \|^2 \geqslant \| x_0 \|^2, \end{aligned}$$

i.e.

$$2\lambda\langle y - x_0 , x_0\rangle + \lambda^2 \| y - x_0 \|^2 \geqslant 0 \qquad \text{for all } 0 \leqslant \lambda \leqslant 1$$

and that $\langle y - x_0 , x_0\rangle \geqslant 0$ as the first term dominates the sign of the total for small values of λ. We have

$$\| y \|^2 = \| y - x_0 + x_0 \|^2 = \| y - x_0 \|^2 + \| x_0 \|^2 + 2\langle y - x_0 x_0\rangle,$$

i.e.

$$\| y \|^2 - \| y - x_0 \|^2 \geqslant \| x_0 \|^2 > 0.$$

We therefore have $\| y \| > \| y - x_0 \|$. In particular $d(x_0 , C_i) < d(0, C_i) < v$. This contradicts the fact that v is the value of the game. Hence the last assertion.

Remark. If C_i's are considered the faces of an n-simplex S with interior in R^n the above theorem proves that we can inscribe a sphere in any norm that touches all the $(n - 1)$ dimensional faces of S. It can also be shown that the sphere is unique. In this connection we would like to mention a theorem of Ky Fan which sharpens the theorem of Berge. Corollary 1 in [3] implies the following result:

In a topological vector space E, let $C_1 , C_2 ,\ldots, C_p$ be closed convex sets such that every $(p - 1)$ of them have nonnull intersection but all of them have empty intersection. If D is a closed set in E such that $D \cup C_1 \cup \cdots \cup C_p$ is convex, then for every nonempty subset I of the set $\{1, 2,\ldots, p\}$, we have

$$D \cap \Big(\bigcap_{i \notin I} C_i\Big) \cap \Big(\bigcap_{i \in I} C_i'\Big) \neq \phi,$$

where $C_i' = E \backslash C$. This result specializes to a theorem of Ghoula–Houri [4] when I contains only one index. It would be interesting if one could prove the above theorem by our game theoretic method.

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